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NORTHWESTERN UNIV EVANSTON IL TECHNOLOGICAL INST
CRITICAL DAMPING IN LINEAR DISCRETE DYNAMIC SYSTEMS. (U)
NOV 79 D E BESKOS, B A BOLEY
TR-1979-2

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REPORT DOCUMENTATION PAGE

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BEFORE COMPLETING FORM

1. REPORT NUMBER 1979-2	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) CRITICAL DAMPING IN LINEAR DISCRETE DYNAMIC SYSTEMS	5. TYPE OF REPORT & PERIOD COVERED 9 Technical Report	
6. PERFORMING ORG. REPORT NUMBER		
7. AUTHOR(s) D. E. Beskos, Dept. of Civil & Mineral Eng. University of Minnesota B. A. Boley, Dean, The Technological Institute	8. CONTRACT OR GRANT NUMBER(s) 15 N00014-75-C-1042	
9. PERFORMING ORGANIZATION NAME AND ADDRESS Northwestern University Evanston, Illinois 60201	10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS NR-064-401 12	
11. CONTROLLING OFFICE NAME AND ADDRESS OFFICE OF NAVAL RESEARCH Arlington, Virginia 22217	12. REPORT DATE 11 November 1979	
	13. NUMBER OF PAGES 21	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) OFFICE OF NAVAL RESEARCH Chicago Branch Office 536 South Clark St. Chicago, Illinois 60605	15. SECURITY CLASS. (of this report) Unclassified	
16. DISTRIBUTION STATEMENT (of this Report) DISTRIBUTION STATEMENT A Approved for public release; Distribution Unlimited		17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report) 14 15-17-19
18. SUPPLEMENTARY NOTES		
19. KEY WORDS (Continue on reverse side if necessary and identify by block number) damping vibrations		
20. ABSTRACT (Continue on reverse side if necessary and identify by block number) Free viscously damped vibrations of linear discrete structural systems are studied. The amount of damping varies among the various structural elements of the system resulting in several critical damping possibilities. A general method is developed for determining the "critical damping surfaces" of a system. These surfaces represent the loci of combinations of damping values corresponding to critically damped motions, and thus separate regions of partial or complete underdamping for those of overdamping. The dimension of a critical damping surface		

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CRITICAL DAMPING IN LINEAR DISCRETE DYNAMIC SYSTEMS

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Abstract

Free viscously damped vibrations of linear discrete structural systems are studied. The amount of damping varies among the various structural elements of the system resulting in several critical damping possibilities. A general method is developed for determining the "critical damping surfaces" of a system. These surfaces represent the loci of combinations of damping values corresponding to critically damped motions, and thus separate regions of partial or complete underdamping from those of overdamping. The dimension of a critical damping surface is equal to the number of independent amounts of damping present in the system. The determination of the surface point corresponding to equal amounts of damping is considerably simplified for systems which, on the assumption that all amounts of damping are equal, possess a damping matrix of the Rayleigh type. Three examples presented in detail illustrate the proposed technique and some of the important characteristics of critical damping surfaces.

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Introduction

The amount of damping in a linear structural dynamic system is usually expressed as a percentage of the critical or the modal critical damping (e.g. [1]). It is then easy to determine if the structure is underdamped or overdamped and consequently to control the response by appropriately varying the amount of structural damping. However, this procedure is possible only when the damping is the same everywhere in the structure or is given in modal form, in which cases the problem of determining the amount of critical damping is essentially one-dimensional and presents no difficulties.

However, for a linear structural dynamic system with viscous damping varying among its elements the problem of determining critical damping becomes much more difficult since there are many critical and partially critical damping possibilities corresponding to different critical element damping value combinations. The importance of this problem lies in the fact that introduces the possibility of controlling the dynamic response more easily and with a greater flexibility by differently varying the damping of a number of elements.

Critical damping analyses are usually performed in connection with free vibration studies which aim at determining natural frequencies and modal shapes. Necessary and sufficient conditions under which discrete damped linear dynamic systems possess classical normal modes have been established by Caughey [2] and Caughey and O'Kelly [3].

For a general discrete viscously damped dynamic system with n degrees of freedom, the free vibration problem primarily consists of determining the roots of the determinantal or characteristic equation of the system. These can be real, complex or purely imaginary and thus characterize the motion as overdamped, underdamped or undamped, respectively. This is what is usually mentioned in the literature (e.g. [1]) without consideration of the case of

critical damping or the case of coexistence of all kinds of the above roots. The only exception appears to be Crafton [4] who considers two degrees of freedom spring-mass systems with two different amounts of viscous damping, and determines the (in general) complex roots of the characteristic equation, omitting, however, a critical damping analysis. The complex roots are obtained by rather cumbersome conformal mapping techniques in [4], but even if more efficient numerical techniques are employed (e.g. [5]), this method of characterization of the motion is not convenient for design purposes, since it requires a complete root determination for every combination of element damping values assumed. Bishop and Johnson [6], on the basis of a particular example of a two degrees of freedom system with only one viscous damper, studied the behavior of the frequencies of the system as damping increased from zero to infinity.

The present paper presents a detailed study of the effect of damping distributed in an arbitrary manner throughout the structural elements on the free motion of the structure. A general method is first proposed to determine the "critical damping surfaces" of a viscously damped linear discrete dynamic system; these are the loci, in "damping space," of amounts of damping leading to critically damped motions. Three examples are then presented in detail which illustrate the proposed method and lead to some interesting and unexpected results.

Free Vibrations of Damped Discrete Systems

Consider a viscously damped linear discrete system of n degrees of freedom in free motion governed by the matrix equation

$$[M]\{\ddot{x}\} + [C]\{\dot{x}\} + [K]\{x\} = \{0\}, \quad (1)$$

where $[M]$, $[C]$ and $[K]$ are symmetric inertia, damping and stiffness $n \times n$ matrices, respectively, $\{x\} = \{x(t)\}$ is the $n \times 1$ displacement vector and dots indicate differentiation with respect to time t .

Assuming a solution of the form

$$\{x\} = \{\phi\}e^{\lambda t}, \quad (2)$$

one reduces (1) to the eigenvalue problem

$$[\lambda^2[M] + \lambda[C] + [K]]\{\phi\} = [D(\lambda)]\{\phi\} = \{0\}, \quad (3)$$

which has nontrivial solutions if, and only if,

$$\det[D(\lambda)] = |D(\lambda)| = 0. \quad (4)$$

Equation (4) is the characteristic or determinantal equation, an algebraic equation of order $2n$ in λ . The roots of (4) can be real, imaginary or complex and occur in pairs of the general form

$$\lambda = -\mu \pm \psi, \quad (5)$$

where μ is nonnegative real, with its negative sign indicating stability of the system, and ψ is a nonnegative real or imaginary number. A real root λ , which should be negative for stability, indicates either overdamping (for which $\psi < \mu$), or critical damping (for which $\psi = 0$); both cases correspond to aperiodic decaying motions. A complex root λ , for which ψ is an imaginary number of the form $j\omega$ ($j = \sqrt{-1}$), indicates underdamping corresponding to an oscillatory decaying motion with a damped natural frequency ω . For an undamped system for which $[C] = [0]$, all the roots of (4) are given by (5) with $\mu = 0$ and $\psi = j\omega_0$ and correspond to periodic motions of natural frequencies ω_0 . If some (all) of the roots of (4) correspond to overdamping, underdamping or critical damping, the system is called partially (completely) overdamped,

underdamped or critically damped, respectively.

Once the roots λ_z ($z = 1, 2, \dots, 2n$) of (4) have been determined, one can solve (3) for the modal shapes $\{\phi\}_z$ which are, in general, complex. Thus, the solution of (1) takes the general form

$$\{x\} = [\phi] \{Qe^{\lambda t}\} \quad (6)$$

where $[\phi]$ is the $n \times 2n$ modal matrix and Q are arbitrary real or complex constants to be determined from the initial conditions of the problem. A necessary and sufficient condition for a passive viscously damped linear discrete system to possess classical normal modes, i.e., a complete set of real orthonormal eigenvectors, is that [3]

$$[C][M]^{-1}[K] = [K][M]^{-1}[C] , \quad (7)$$

in which case these classical normal modes are identical with the normal modes of the undamped system.

Critical Damping Surfaces

Consider a n degrees of freedom linear damped dynamic system characterized by the matrices $[M]$, $[K]$, and $[C]$, where the elements of $[C]$ are combinations of the m different damping coefficients c_k ($k = 1, 2, \dots, m$) of the various elements of the system. The number m can be greater, equal or smaller than n . For partial (complete) overdamping or critical damping some (all) of the roots of (4) are of the form

$$\lambda = -b, \quad b > 0 \quad (8)$$

and in that case (4) becomes

$$|b^2[M] - b[C] + [K]| = |D(b, c_k)| = 0. \quad (9)$$

In the m -dimensional space with coordinates c_k ($k = 1, 2, \dots, m$), equation (9) represents a family of m -dimensional surfaces S_b . Each member of the family corresponds to overdamping or critical damping and is characterized by its own value of b , which is a function of the c_k 's. The problem is to determine that b which corresponds to the "critical damping surface," i.e., the locus of combinations of c_k leading to critically damped motion and thus separating regions of partial or complete underdamping from those of overdamping. There are actually q ($q \leq n$) critical damping surfaces since there are at most as many partial critical damping possibilities as the number of the pairs of roots λ in (5) with $\psi = 0$. Since critical damping represents the threshold between overdamping and underdamping, one can conclude that among the S_b surfaces, the critical surface S_{cr} is the one for which the damping will be a minimum, i.e.,

$$\left. \begin{aligned} (d/db)|D(b, c_k)| &= (\partial/\partial b)|D(b, c_k)| + \sum_k (\partial/\partial c_k)|D(b, c_k)|(\partial c_k/\partial b) = 0, \\ (\partial c_k/\partial b) &= 0, \quad k = 1, 2, \dots, m. \end{aligned} \right\} \quad (10)$$

An alternative derivation of (9) and (10) is the following: Consider equation (4) with $\lambda = -R + jI$, ($R > 0$, $I = \text{real}$, $j = \sqrt{-1}$), i.e.,

$$\Delta(\lambda) = \Delta(-R, jI) = |D(-R + jI)| = 0. \quad (11)$$

Expansion of $\Delta(\lambda)$ in Taylor series about the point $\lambda_0 = (-R, 0)$ yields

$$\begin{aligned} \Delta(-R, jI) = & \Delta(-R, 0) + (jI) \frac{\partial \Delta}{\partial \lambda} \Big|_{(-R, 0)} + \frac{1}{2!} (-I^2) \frac{\partial^2 \Delta}{\partial \lambda^2} \Big|_{(-R, 0)} + \\ & + \frac{1}{3!} (-jI^3) \frac{\partial^3 \Delta}{\partial \lambda^3} \Big|_{(-R, 0)} + \dots, \end{aligned} \quad (12)$$

which in view of (11) reduces to

$$\left. \begin{aligned} \Delta(-R, 0) - \frac{1}{2!} I^2 \frac{\partial^2 \Delta}{\partial \lambda^2} \Big|_{(-R, 0)} + \dots &= 0, \\ \frac{\partial \Delta}{\partial \lambda} \Big|_{(-R, 0)} - \frac{1}{3!} I^2 \frac{\partial^3 \Delta}{\partial \lambda^3} \Big|_{(-R, 0)} + \dots &= 0. \end{aligned} \right\} \quad (13)$$

For critical damping, for which $I = 0$, equations (13) lead to equations

$$\left. \begin{aligned} \Delta(-R, 0) &= 0, \\ (\partial \Delta / \partial \lambda) \Big|_{(-R, 0)} &= 0, \end{aligned} \right\} \quad (14)$$

which are the same as (9) and (10).

In principle, one can solve (10) for b and obtain its critical value b_{cr} as a function of the c_k 's. Thus, the equation of critical damping surfaces will be given by (9) with $b = b_{cr}$, i.e., by

$$|b_{cr}^2[M] - b_{cr}[C] + [K]| = 0. \quad (15)$$

In practice, however, the nonlinear system of (9) and (10) has to be solved numerically. Thus, provided that differentiations in (10) can be done analytically, one determines numerically a finite number of groups of $m+1$ values for b_{cr} and c_{kcr} satisfying (9) and (10) simultaneously, and one is then able to represent critical damping surfaces in the m -dimensional c_k space as sets of points with coordinates c_{kcr} . Eventhough the above method of determining critical damping surfaces is quite general and applicable, in principle, to a n degrees of freedom system, the fact that, to the authors knowledge, there is no presently available efficient numerical treatment of the differentiations in (10), limits the range of applicability of the method to small order systems.

However, the particular point $c_1 = c_2 = \dots = c_m = c$ of a critical damping

surface can be very easily obtained for the special case in which $[C]_{c_k=c}$ is of the Rayleigh type, i.e., of the form

$$[C]_{c_k=c} = a_1 [M] + a_2 [K] , \quad (16)$$

where a_1 and a_2 are constants. For this case (3) becomes

$$[(\lambda^2 + a_1 \lambda) [M] + (1 + a_2 \lambda) [K]] \{\phi\} = \{0\} , \quad (17)$$

indicating that there exist classical normal modes and that for the i_{th} ($i = 1, 2, \dots, n$) mode with underdamping

$$(\lambda_i^2 + a_1 \lambda_i) / (1 + a_2 \lambda_i) = -w_{oi}^2 . \quad (18)$$

Equation (18) can be solved for λ_i and give

$$\lambda_i = -b_i \pm j w_i , \quad (19)$$

where

$$\left. \begin{aligned} b_i &= 1/2(a_1 + a_2 w_{oi}^2) , \quad w_i^2 = w_{oi}^2 - b_i^2 , \\ a_1 + a_2 w_{oi}^2 &\geq 0 , \quad j = \sqrt{-1} . \end{aligned} \right\} \quad (20)$$

It is apparent then, in view of (19) and (20), that at critical damping one has $w_i = 0$, or

$$b_{icr} = w_{oi} , \quad (21)$$

with a_1 and a_2 satisfying the relation

$$a_{2cr} w_{oi}^2 - 2w_{oi} + a_{1cr} = 0 , \quad (22)$$

where a_{1cr} and a_{2cr} are their values at critical damping. Equations (21) and (22)

therefore represent the solution of (10) for the particular surface point

$c_1 = c_2 = \dots = c_m = c$ and for systems obeying (16). This solution is unique if one of the a_1 and a_2 is zero. Once the b_{icr} 's have been computed from (21), a numerical evaluation of the left hand side of (15) for a sequence of values of c leads to the determination of the common value c of the c_k 's which satisfies (15) for every i . One important result of this analysis is that, as equation (21) clearly demonstrates, for systems obeying (16), one has $q=n$, i.e., as many

critical damping surfaces as there are degrees of freedom in the system.

Three illustrative examples follow.

Example 1

For the familiar one degree of freedom spring-dashpot-mass system the equation of free motion is

$$\ddot{x} + 2\beta\dot{x} + w_0^2 = 0, \quad (23)$$

where the measure of the amount of damping β and the undamped natural frequency w_0 are given in terms of the mass m , the stiffness K and the damping coefficient c of the system as

$$\beta = c/2m, \quad w_0 = \sqrt{K/m}. \quad (24)$$

Here $m=n=1$ so that there exists only one critical damping surface, of dimension one, i.e., a point. The condition (9) for overdamping or critical damping is

$$b^2 - 2b\beta + w_0^2 = 0. \quad (25)$$

Use of (10) provides $b_{cr} = \beta_{cr}$ which, in conjunction with (25), leads to

$$\beta_{cr} = w_0, \quad (26)$$

which is the well known condition of critical damping for this system. Notice that (26) could also have been obtained from (20), (21) and (24) by realizing that $c = (c/m)m = a_1 m$, in which case $b_{cr} = (1/2)a_{1cr} = c_{cr}/2m = \beta_{cr} = w_0$.

Example 2

Consider the two degrees of freedom system of Fig. 1 consisting of a variable torsional stiffness shaft with two flywheels and one viscous damper and being in free torsional vibration. This system has been taken from reference [6] (p. 507) which studies the way in which the free motion of the system changes as the damping is increased from zero to infinity by determining the roots of the determinantal equation for a sequence of values of the amount of damping present. Here the results of reference [6] will be verified by applying the

proposed method for analyzing damped systems.

The equation of free vibration of the system of Fig. 1 is

$$\begin{bmatrix} 2I & 0 \\ 0 & 2I \end{bmatrix} \begin{Bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2c \end{bmatrix} \begin{Bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{Bmatrix} + \begin{bmatrix} 6K & -4K \\ -4K & 6K \end{bmatrix} \begin{Bmatrix} \theta_1 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad (27)$$

where I , c and K stand for the mass moment of inertia of the flywheel, the damping coefficient, and the torsional stiffness of the shaft, respectively, and θ_1 and θ_2 represent the two rotational degrees of freedom of the system. Equation (27) clearly indicates that conditions (7) and (16) are not satisfied. Thus, this system does not possess classical normal modes and the only way to determine its critical damping surfaces is through the general method.

The determinantal equation (4) for this system, in view of (27), takes the form

$$\Lambda^4 + 2\alpha\Lambda^3 + 6\Lambda^2 + 6\alpha\Lambda + 5 = 0, \quad (28)$$

where

$$\Lambda = \lambda/w_1, \quad \alpha = c/2Iw_1, \quad w_1^2 = K/I. \quad (29)$$

For the case of overdamping or critical damping for which $\Lambda = -B = -b/w_1$, (28) becomes

$$B^4 - 2\alpha B^3 + 6B^2 - 6\alpha B + 5 = 0. \quad (30)$$

Differentiating (30) with respect to B and requiring that $(d\alpha/dB) = 0$, one obtains from (30)

$$\alpha = 2B(3+B^2)/3(1+B^2). \quad (31)$$

With (31), (30) takes the form

$$B^6 + 3B^4 + 3B^2 - 15 = 0. \quad (32)$$

The only one real positive root of (32) is $B = 1.233$ for which (31) yields the equation of the only existing here critical damping surface - a point, as

$$\alpha_{cr} = 1.474, \quad (33)$$

a value which is identical to that obtained in [6].

The existence of just one critical damping surface for this two degree of freedom system indicates that this actually corresponds to partial critical

damping. For $0 < \alpha < 1.474$ the system is underdamped, for $\alpha = 1.474$ it is partially critically damped, and for $\alpha > 1.474$ it is partially underdamped. Thus, no matter how great the damping parameter α may be, there is no way to reach complete overdamping, exactly because there is no other critical damping surface for $\alpha > 1.474$. This observation was also made in [6] by studying the behavior of the roots of (28) as the damping parameter α was increasing. In fact, it was also shown in [6] that for α approaching to infinity, (28) possesses only two imaginary conjugate roots which is physically explained by the fact that at $\alpha \rightarrow \infty$ the damped flywheel becomes locked and the other flywheel, with its two shafts, forms a one degree of freedom conservative system.

This example clearly demonstrates the simplicity of the proposed method for determining critical damping. In comparison reference [6] required a laborious numerical computation and detailed plot of the real and imaginary parts of the four roots of equation (28) for a lengthy sequence of values of the damping parameter α .

Example 3

Consider the two degrees of freedom spring-dashpot-mass system shown in Fig. 2, where K_i , c_i and m_i ($i = 1, 2$) stand for spring constants, coefficients of viscous damping and masses, respectively, and x_1 and x_2 represent the two translational degrees of freedom of the system. Notice that for $c_1 = 0$ one has a case similar to that treated in Example 2. The equations of free motion of this system are

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} c_1 & 0 \\ 0 & c_2 \end{bmatrix} \begin{Bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{Bmatrix} + \begin{bmatrix} K_1 + K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \quad (34)$$

For this system, equation (4) becomes

$$\lambda^4 + 2(\beta_1 + \beta_2)\lambda^3 + (w_1^2 + w_2^2 + w_{12}^2 + 4\beta_1\beta_2)\lambda^2 + 2(\beta_1w_2^2 + \beta_2w_1^2 + \beta_2w_{12}^2)\lambda + w_1^2w_2^2 = 0, \quad (35)$$

where

$$\left. \begin{aligned} w_1^2 &= K_1/m_1, \quad w_2^2 = K_2/m_2, \quad w_{12}^2 = K_{12}/m_1, \\ \beta_1 &= c_1/2m_1, \quad \beta_2 = c_2/2m_2, \end{aligned} \right\} \quad (36)$$

while equation (9) takes the form

$$b^4 - 2(\beta_1 + \beta_2)b^3 + (w_1^2 + w_2^2 + w_{12}^2 + 4\beta_1\beta_2)b^2 - 2(\beta_1w_2^2 + \beta_2w_1^2 + \beta_2w_{12}^2)b + w_1^2w_2^2 = 0. \quad (37)$$

The conditions of critical damping described by (10) yield

$$b^3 - (3/2)(\beta_1 + \beta_2)b^2 + (1/2)(w_1^2 + w_2^2 + w_{12}^2 + 4\beta_1\beta_2)b - (1/2)(\beta_1w_2^2 + \beta_2w_1^2 + \beta_2w_{12}^2) = 0. \quad (38)$$

Thus, the critical damping surfaces of this system are curves in the (β_1, β_2) plane described by (37) with $b = b_{cr}$ being a function of β_1 and β_2 which can be obtained by solving (38). Construction of these curves can, in general, be accomplished by computing the left hand sides of (37) and (38) for all possible combinations of a finite number of values for β_1 , β_2 and b , and selecting those combinations that simultaneously satisfy (37) and (38); these combinations provide the points $(\beta_{1cr}, \beta_{2cr})$ of the critical damping curves. In this case, however, the construction of the critical damping curves can be done more easily by utilizing well known formulae (e.g. [7] p. 86) to solve explicitly the cubic algebraic equation (38) for b in terms of β_1 and β_2 . In general, equation (38) may have one real root or three unequal real roots depending on the values of β_1 and β_2 . In this problem one is interested in positive roots of (38) which upon substitution into (37) satisfy it for real positive values of β_1 and β_2 . One can very easily prove by utilizing (37) that the slope $d\beta_2/d\beta_1$ of the curves at the points $\beta_1 = \beta_2$, $\beta_1 = 0$ and $\beta_2 = 0$ is negative.

Fig. 3 shows the two critical damping curves C_1 and C_2 of the two degrees of freedom system of Fig. 2 in the (β_1, β_2) plane for the following numerical

data:

$$\left. \begin{aligned} K_1 &= 3000 \text{ lb/ft} = 43,779.528 \text{ N/m}, \\ K_2 &= 4000 \text{ lb/ft} = 58,372.703 \text{ N/m}, \\ m_1 &= m_2 = 1 \text{ lb}\cdot\text{sec}^2/\text{ft} = 14.593 \text{ N sec}^2/\text{m}, \\ &\quad (1 \text{ ft} = 0.3048 \text{ m}, 1 \text{ lb} = 4.448 \text{ N}). \end{aligned} \right\} \quad (39)$$

The two critical damping curves were constructed with the aid of a computer by both the purely numerical method and the one based on the solutions of the cubic equation. It was found that for the curve C_1 the cubic equation had only one real root, while for the curve C_2 it had three real roots from which only one was acceptable. For $\beta_1 = \beta_2 = \beta$, condition (16) is satisfied with

$$\alpha_1 = 2\beta, \quad \alpha_2 = 0 \quad (40)$$

and thus (21), (22) and (40) lead to

$$b_{icr} = \beta_{cr} = w_{oi}, \quad (i = 1, 2) \quad (41)$$

where the w_{oi} 's are the positive real roots of the frequency equation

$$w_0^4 - (w_1^2 + w_2^2 + w_{12}^2) w_0^2 + w_1^2 w_2^2 = 0, \quad (42)$$

which can be obtained from (35) for $\beta_1 = \beta_2 = 0$ and $\lambda = jw_0$. For the data of (39), equation (42) yields

$$w_{o1} = 24.78 \text{ rad/sec}, \quad w_{o2} = 69.90 \text{ rad/sec}. \quad (43)$$

The satisfaction of (16) also verifies the fact that $q = n = 2$ for this system.

The critical damping curves C_1 and C_2 divide the plane (β_1, β_2) into the regions R_1, R_2 and R_3 , as shown in Fig. 3. Curve C_1 , for which $\beta_1 = \beta_2 = w_{o1} = 24.78$, defines a state of partial critical damping and separates a region of complete underdamping (region R_1) from a region of partial underdamping (region R_2). Curve C_2 , for which $\beta_1 = \beta_2 = w_{o2} = 69.90$, defines a state of complete critical damping and separates a region of partial underdamping (region R_2) from a region of complete overdamping (region R_3). The results of Fig. 3 were also verified numerically by determining on the computer all the four roots of equation (35)

with the aid of Newton's iterative method (e.g. [5]) for all the possible combinations of numerical values of β_1 and β_2 ranging from 0 to 120 in steps of 5. Indeed, the results indicated that all the roots in region R_1 were complex conjugate (complete underdamping), all the roots in region R_3 were negative real (complete overdamping), while two roots were negative real and two complex conjugate in region R_2 (partial underdamping). Furthermore, for the point $\beta_1 = \beta_2 = \beta$ on the curve C_1 there were found two complex conjugate roots and two negative real roots both equal in magnitude to $w_{01} = 24.78$ (partial critical damping), while for the point $\beta_1 = \beta_2 = \beta$ on the curve C_2 there were found two negative real roots both equal in magnitude to $w_{02} = 69.90$ and two more unequal negative real roots (complete critical damping).

The interesting thing in this example is the fact that region R_2 is unbounded and that even for values of β_1 or β_2 (but not both) approaching infinity there is still partial underdamping, while one might expect to achieve overdamping if one of the β 's is large enough. This phenomenon is the two-dimensional counterpart of that of Example 2 and is amenable to an analogous physical explanation. Indeed, at the limit, as β_1 , for example, approaches infinity, the mass m_1 becomes essentially locked, x_1 approaches zero, the system becomes a single degree of freedom one, and (34) reduces to

$$\ddot{x}_2 + 2\beta_2 \dot{x}_2 + w_2^2 x_2 = 0, \quad (44)$$

for which the condition for overdamping or critical damping is

$$b^2 - 2\beta_2 b + w_2^2 = 0. \quad (45)$$

Use of (10) then yields

$$\beta_{2cr} = w_2 = 44.72. \quad (46)$$

Similarly, at the limit, as β_2 approaches infinity, mass m_2 becomes locked, x_2 approaches zero, the system becomes a single degree of freedom one, and (34) reduces to

$$\ddot{x}_1 + 2\beta_1 \dot{x}_1 + (w_1^2 + w_{12}^2) x_1 = 0, \quad (47)$$

for which finally (10) yields

$$\beta_{1cr} = \sqrt{w_1^2 + w_{12}^2} = 59.16 . \quad (48)$$

Equations (46) and (48) are equations for the asymptotes of the critical damping curves and indicate that even for infinite values of β_1 (β_2) there are values of β_2 (β_1) in region R_2 , which are less than their critical values, for which there is partial underdamping.

During the construction of the critical damping curves C_1 and C_2 it was observed that the values of b_{1cr} and b_{2cr} were very close to w_{01} and w_{02} , respectively, for any point (β_1, β_2) on the curves. This means that one can approximately construct the curves C_1 and C_2 by assuming that (41) holds true not only for the points $\beta_1 = \beta_2 = \beta$ of the curves but for any point of them. Thus, (37) with $b_{icr} = w_{0i}$ finally becomes

$$\beta_2 = \frac{A_i \beta_1 - B_i}{E_i \beta_1 - D_i} , \quad (49)$$

where

$$\left. \begin{aligned} A_i &= 2w_{0i} (w_2^2 + w_{0i}^2) , \\ B_i &= w_{0i}^4 + w_1^2 w_2^2 + w_{0i}^2 (w_1^2 + w_2^2 + w_{12}^2) , \\ E_i &= 4w_{0i} \\ D_i &= 2w_{0i} (w_1^2 + w_{12}^2 + w_{0i}^2) , \quad i = 1, 2 . \end{aligned} \right\} \quad (50)$$

It is evident from (49) that the approximate critical damping curves are two equilateral hyperbolas with two branches each. These branches are symmetric about the intersection point of their asymptotes $\beta_1 = D_i/E_i$ and $\beta_2 = A_i/E_i$. Out of the two branches constituting every hyperbola only one branch, that for which $\beta_1 = \beta_2 = w_{0i}$, represents a critical damping curve. It is easy to determine the sign of the slope of these branches by computing $d\beta_2/d\beta_1$ from (49) and utilizing relations (50). Thus, one has

$$(d\beta_2/d\beta_1) = -4w_{0i}^2 w_2^2 w_{12}^2 / (E_i \beta_1 - D_i)^2 , \quad (51)$$

indicating that the slope is negative everywhere.

A plot of the approximate critical damping curves on the basis of the numerical data (39) is shown in Fig. 3. The approximate curves of course coincide with the exact curves at $\beta_1 = \beta_2$, and they slightly depart from them more and more as the β 's increase. Such approximate critical damping curves can be very useful for rapid design calculations, although, unfortunately, no general extensions of their validity can be presented here.

Conclusions

The following remarks can be made on the basis of the preceding developments about the free vibrations of linear discrete structural systems with viscous damping varying among their elements:

- 1) There exist "critical damping surfaces" for every system representing the loci of combinations of damping leading to partial or complete critically damped motion and thus separating regions of partial or complete underdamping from those of overdamping.
- 2) A general method is proposed for determining the equations of these critical damping surfaces. The determination of the surface point corresponding to equal amounts of damping is considerably simplified for systems which, on the assumption that all amounts of damping are equal, possess a damping matrix of the Rayleigh type.
- 3) The dimension of these critical damping surfaces is equal to the number of independent damping values in the system, while their number is less or equal than the number of the system degrees of freedom. For a n degrees of freedom system with a damping matrix reducible to Rayleigh type for equal amounts of damping, there are at most one completely underdamped region, one completely overdamped region and $n-2$ partially underdamped regions.
- 4) More research is needed in connection with the determination and identification of the critical damping surfaces for large order systems which require an efficient numerical treatment.

Acknowledgements

The authors acknowledge the support of the University of Minnesota Computer Center. The second author also acknowledges the support of the Office of Naval Research.

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Captions of the Figures

Figure 1: The two degrees of freedom system of Example 2.

Figure 2: The two degrees of freedom system of Example 3.

Figure 3: Critical damping curves for the system of Figure 2.

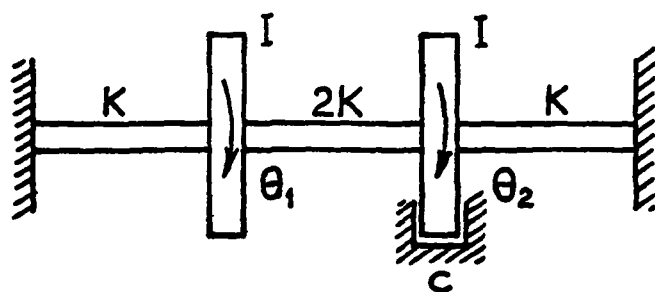


Figure 1

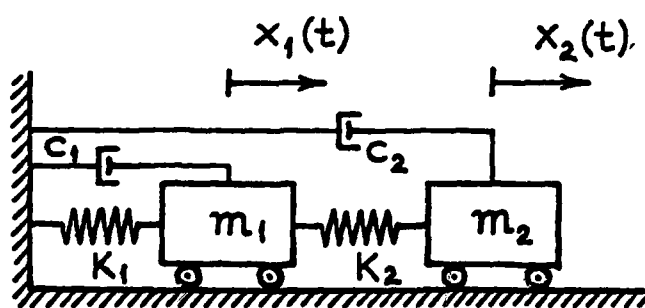


Figure 2

Figure 3.

